

Lyapunov exponents for products of complex Gaussian random matrices

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Abstract

The exact value of the Lyapunov exponents for the random matrix product $P_N = A_N A_{N-1} \cdots A_1$ with each $A_i = \Sigma^{1/2} G_i^c$, where Σ is a fixed $d \times d$ positive definite matrix and G_i^c a $d \times d$ complex Gaussian matrix with entries standard complex normals, are calculated. Also obtained is an exact expression for the sum of the Lyapunov exponents in both the complex and real cases, and the Lyapunov exponents for diffusing complex matrices.

1 Introduction

Presently there is a great deal of interest, from the viewpoints of probability theory and applications to physics and communications engineering, in the statistical properties of large random matrices (see e.g. the recent texts [1, 22, 12, 29]). Looking back to the mid 1980's it would have been fair to say that the same applied to the topic of products of $d \times d$ random matrices. Thus it was in that era that the foundational probabilistic works of Kesten, Furstenberg, Oseledec and others in the 1960's and 70's had matured to the extent that a book on the subject was written [4]; that a summer research conference was held with this topic dominating the subsequent proceedings [7]; and that a number of applications to physics were in the full swing of investigation, culminating in the appearance of the research monograph [10].

There have been some present day works that have aimed to combine the contemporary interest in the eigenvalues of large random matrices with the topic of products of random matrices, by studying eigenvalue distributions of products of random matrices, in the limit that the size of the matrices is large [5, 6, 23, 14, 20, 28]. There has also been a good deal of present day activity relating to the numerical computation of Lyapunov exponents [31, 2, 11, 3, 30, 24]. Regarding the latter, let

$$P_N = A_N A_{N-1} \cdots A_1, \quad (1.1)$$

where each A_i is a $d \times d$ independent, identically distributed random matrix such that the diagonal elements of $A^\dagger A$ have finite second moments. According to the multiplicative ergodic theorem of Oseledec [21, 25], one has that the limiting matrix

$$V_d := \lim_{N \rightarrow \infty} (P_N^\dagger P_N)^{1/(2N)} \quad (1.2)$$

is well defined, with d positive real eigenvalues $e^{\mu_1} \geq e^{\mu_2} \geq \dots \geq e^{\mu_d}$. The $\{\mu_i\}$ are referred to as the Lyapunov exponents. Already in 1973 Kingman had nominated methods to compute the largest Lyapunov exponent [15] as an outstanding problem in the field. The work [24] solves this problem in the case when each A_i is chosen from a finite set, and has positive entries.

Implicit in the need for efficient computational methods is that it is generally not possible to compute the Lyapunov exponents analytically. Some noteworthy exceptions occur in the case $d = 2$; see e.g. [16, 17, 9] and references therein. For general d , apart from the case of diagonal matrices, it seems that the only exact computation of the Lyapunov exponents recorded in the literature is when the A_i are real Gaussian matrices with entries independent standard real normals. Then it is a result of Newman [19] that

$$\mu_i = \frac{1}{2} \left(\log 2 + \Psi \left(\frac{d-i+1}{2} \right) \right) \quad (i = 1, \dots, d), \quad (1.3)$$

where $\Psi(x)$ denotes the digamma function.

It is the aim of this paper to extend available exact results on the evaluation of Lyapunov exponents for certain $d \times d$ random matrices. In particular, a closed form evaluation of $\{\mu_i\}$ is obtained for

$$A_i = \Sigma^{1/2} G_i^c, \quad (1.4)$$

where Σ is a fixed $d \times d$ positive definite matrix and G_i^c a $d \times d$ complex Gaussian matrix with entries standard complex normals. Partial results are also obtained relating to the exact evaluation of $\{\mu_i\}$ for $A_i = \Sigma^{1/2} G_i^r$, where G_i^r denotes a $d \times d$ real Gaussian with independent standard normals as entries. The latter supplements the $\Sigma = I_d$ result (1.3), and the $d = 2$ result

$$\mu_1 = -\frac{1}{2}\gamma + \frac{1}{2} \log \left(\frac{1}{2} \text{Tr } \Sigma + \sqrt{\det \Sigma} \right), \quad (1.5)$$

where γ denotes Euler's constant.

Our first result is the analogue of (1.3) for complex Gaussian matrices with entries independent standard complex normals.

Proposition 1. *Consider the matrix product (1.1), with each A_i given by (1.4) with $\Sigma = I_d$. We have*

$$\mu_i = \Psi(d - i + 1). \quad (1.6)$$

This result can be generalized by allowing for general Σ in (1.4).

Proposition 2. Consider the matrix product (1.1), with each A_i given by (1.4) for general positive definite Σ . Let the eigenvalues of Σ^{-1} be denoted $\{y_j\}_{j=1,\dots,d}$. We have

$$\mu_k = -\frac{1}{2\prod_{1\leq i<j\leq d}(y_j - y_i)} \det \begin{bmatrix} [y_j^{i-1}]_{i=1,\dots,k-1}^{j=1,\dots,d} \\ [(\log y_j)y_j^{k-1}]_{j=1,\dots,d} \\ [y_j^{i-1}]_{i=k+1,\dots,d}^{j=1,\dots,d} \end{bmatrix} + \frac{1}{2}\Psi(k). \quad (1.7)$$

Corollary 1. In the setting of Proposition 2, one has the sum rule

$$\mu_1 + \mu_2 + \dots + \mu_d = \frac{1}{2} \sum_{m=1}^d \left(-\log y_m + \Psi(m) \right). \quad (1.8)$$

We remark that for d large, the fact that $\Psi(d) \sim \log d$ tells us that the eigenvalues y_m must have the scaled form $y_m/d = Y(m/d)$, with $Y(0) = 0$ a PDF on $(0, 1)$, for (1.8) to have a well defined average as $d \rightarrow \infty$.

The above results will be proved in the next section. It will furthermore be showed that Corollary 1 can be proved independent of Proposition 2, and this will allow an analogue of (1.8) in the case of real random matrices $A_i = \Sigma^{1/2}G_i^r$ to be derived (see eq. (2.25) below). We give too, in eq. (2.28), the evaluation of the so-called generalized maximum Lyapunov exponent [10]

$$L(q) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left\langle ||P_N||^q \right\rangle \quad (1.9)$$

in the case of (1.4). In section 3 we calculate the Lyapunov exponents for

$$A_i := \lim_{m \rightarrow \infty} \prod_{j=1}^m e^{C_j^{(i)}/m^{1/2}}, \quad C_j^{(i)} = H_1 + iH_2, \quad (1.10)$$

where H_1 and H_2 are Hermitian matrices distributed with density function proportional to $\exp(-\text{Tr}H_1^2/2w_1)$ and $\exp(-\text{Tr}H_2^2/2w_2)$ respectively. Section 4 discusses features of μ_1 as given by (1.7).

2 Proofs

2.1 Background theory

A fundamental characterization of the Lyapunov exponents defined below (1.2) is that they satisfy [21, 25]

$$\mu_1 + \dots + \mu_k = \sup \lim_{N \rightarrow \infty} \frac{1}{N} \log \text{Vol}_k \{y_1(N), \dots, y_k(N)\} \quad (k = 1, \dots, d). \quad (2.1)$$

In (2.1) $y_j(N) := P_N y_j(0)$, the supremum is over all sets of linearly independent vectors $\{y_1(0), \dots, y_k(0)\}$ and Vol_k refers to the (generalized) volume of the parallelogram generated by the given set of k vectors. In regards to the latter, with

$$B_N := [y_1(N) \ y_2(N) \ \dots \ y_k(N)],$$

so that B_N is the $d \times k$ matrix with its columns given by the k vectors $\{y_j(N)\}$, we have

$$\text{Vol}_k\{y_1(N), \dots, y_k(N)\} = \det(B_N^\dagger B_N)^{1/2} = \det(B_0^\dagger P_N^\dagger P_N B_0)^{1/2} \quad (2.2)$$

Following [8, 19] the basic fact that makes the computation of (2.1) tractable for matrices (1.4) is that the distribution of the random vector $G_i^c \vec{x}/|\vec{x}|$, $\vec{x} \neq \vec{0}$, is independent of \vec{x} . Thus with $\{y_1(0), \dots, y_k(0)\}$ a set of k linearly independent unit vectors, and $E_{d \times k}$ denoting the $d \times k$ matrix with 1's in the diagonal positions of the k rows, and 0's elsewhere, we have

$$B_0^\dagger P_N^\dagger P_N B_0 = \prod_{j=1}^N E_{d \times k}^T G_j^{c\dagger} \Sigma G_j^c E_{d \times k} = \prod_{j=1}^N G_{j,k}^{c\dagger} \Sigma G_{j,k}^c, \quad (2.3)$$

where $G_{j,k}^c$ denotes the $d \times k$ matrix formed by the first k columns of G_j^c . Substituting in (2.2), then substituting the result in (2.1), we see firstly that there is no longer any dependence on $\{y_1(0), \dots, y_k(0)\}$, so the sup operation in (2.1) is redundant. We are then left with the expression

$$\mu_1 + \dots + \mu_k = \lim_{N \rightarrow \infty} \sum_{j=1}^N \frac{1}{N} \log \det \left(G_{j,k}^{c\dagger} \Sigma G_{j,k}^c \right)^{1/2}. \quad (2.4)$$

But each G_j^c independently belongs to the set of complex rectangular $d \times k$ Gaussian matrices $\mathcal{N}_{d \times k}^c(0, 1)$ in which each entry is a standard complex normal. The law of large numbers tells us that the limit in (2.4) can be evaluated as an average over this set,

$$\mu_1 + \dots + \mu_k = \left\langle \log \det \left(G_k^{c\dagger} \Sigma G_k^c \right)^{1/2} \right\rangle_{\mathcal{N}_{d \times k}^c(0,1)}. \quad (2.5)$$

2.2 Proof of Proposition 1 and Corollary 1

Proposition 1 relates to the case $\Sigma = I_d$. Now the set of matrices $\mathcal{N}_{d \times k}^c(0, 1)$ have probability density function proportional to $e^{-\text{Tr}(G_j^{c\dagger} G_j^c)}$. Thus the average in (2.5) is a function of $G_j^{c\dagger} G_j^c$. Introducing the complex Wishart matrix $W = G_j^{c\dagger} G_j^c$, we know that the corresponding Jacobian is proportional to $(\det W)^{d-k}$ (see e.g. [12, Eq. (3.23)]). Making use too of the simple identity

$$\frac{d}{d\mu} (\det W)^\mu \Big|_{\mu=0} = \text{Tr} \log \det W \quad (2.6)$$

it is therefore possible to rewrite (2.4) in the case $\Sigma = I_d$ as

$$\mu_1 + \dots + \mu_k = \frac{1}{2} \frac{d}{d\mu} \left\langle (\det W)^{\mu+d-k} e^{-\text{Tr} W} \right\rangle_{W>0} \Big|_{\mu=0}, \quad (2.7)$$

where the average is over all positive definite $k \times k$ complex Hermitian matrices.

We see that (2.7) is a function only of the eigenvalues of W . Changing variables to the eigenvalues and eigenvectors (see e.g. [12, Proposition 1.3.4]) gives

$$\mu_1 + \cdots + \mu_k = \frac{1}{2Z_{d,d-k}} \frac{d}{d\mu} Z_{d,d-k+\mu} \Big|_{\mu=0}, \quad (2.8)$$

where

$$Z_{d,c} := \int_0^\infty dx_1 \cdots \int_0^\infty dx_d \prod_{j=1}^d e^{-x_j} x_j^c \prod_{1 \leq j < l \leq d} (x_l - x_j)^2. \quad (2.9)$$

But the integral $Z_{c,k}$ is a particular limiting case of the Selberg integral and as such has a product of gamma function evaluation (see e.g. [12, Prop. 4.7.3 with $\beta = 2$]), telling us that

$$\begin{aligned} \mu_1 + \cdots + \mu_k &= \frac{1}{2} \frac{d}{d\mu} \prod_{j=0}^{k-1} \frac{\Gamma(d-k+\mu+1+j)}{\Gamma(d-k+1+j)} \\ &= \frac{1}{2} \sum_{j=0}^{k-1} \Psi(d-k+1+j) = \frac{1}{2} \sum_{j=0}^{k-1} \Psi(d-j). \end{aligned} \quad (2.10)$$

The result (1.6) is now immediate.

Let's now consider Corollary 1. We thus want to evaluate (2.5) in the case $k = d$ but with Σ a general positive definite matrix. For $d = k$ we can write $\det(G_k^{c\dagger} \Sigma G_k^c)^{1/2} = \det(G_k^c G_k^{c\dagger} \Sigma)^{1/2}$ since then $G_k^c G_k^{c\dagger}$ has full rank. Furthermore the probability density corresponding to $\mathcal{N}_{d \times k}^c(0, 1)$ is then proportional to $\exp(-\text{Tr } G_k^c G_k^{c\dagger})$. Introducing the complex Wishart matrix $W = G_d^c G_d^{c\dagger}$, for which the corresponding Jacobian is just a constant, we therefore have

$$\mu_1 + \cdots + \mu_d = \frac{1}{2} \left(\log \det \Sigma + \left\langle \log(\det W) e^{-\text{Tr } W} \right\rangle_{W>0} \right).$$

Noting that the latter average is just (2.10) in the case $k = d$ gives (1.8).

2.3 Proof of Proposition 2

Writing $X = \Sigma^{1/2} G_k^c$, (2.5) can be rewritten

$$\mu_1 + \cdots + \mu_k = \left\langle \log \det \left(X^\dagger X \right)^{1/2} \right\rangle_{X \in \Sigma^{-1/2} \mathcal{N}_{d \times k}^c(0, 1)}. \quad (2.11)$$

Now the set of matrices $\Sigma^{-1/2} \mathcal{N}_{d \times k}^c(0, 1)$ have probability density function proportional to $\exp(-\text{Tr}(X X^\dagger \Sigma^{-1}))$. The $d \times d$ matrix $X X^\dagger$ has rank $k \leq d$, and so has $d - k$ zero eigenvalues. This feature, in the case $d < k$, distinguishes the set of matrices $\Sigma^{-1/2} \mathcal{N}_{d \times k}^c(0, 1)$ from $\mathcal{N}_{d \times k}^c(0, 1) \Sigma_k^{-1/2}$. Thus the latter has probability density proportional to $\exp(-\text{Tr}(X^\dagger X \Sigma_k^{-1}))$ with $X^\dagger X$ of full rank and thus having no zero eigenvalue.

These matrices are termed complex Wishart matrices of mean zero and covariance Σ_k , while the matrices XX^\dagger are sometimes referred to as complex pseudo Wishart matrices of mean zero and covariance Σ .

Let $\rho_{(1)}(\lambda; \Sigma)$ denote the eigenvalue density of the nonzero eigenvalues $\{\lambda_j\}_{j=1, \dots, k}$ for the ensemble of complex pseudo Wishart matrices so specified. Noting that

$$\log \det (X^\dagger X)^{1/2} = \frac{1}{2} \sum_{j=1}^k \log \lambda_j \quad (2.12)$$

and so is a linear statistic in $\{\lambda_j\}$, it follows from (2.11)

$$\mu_1 + \dots + \mu_k = \frac{1}{2} \int_0^\infty (\log \lambda) \rho_{(1)}(\lambda; \Sigma) d\lambda. \quad (2.13)$$

Studies in wireless communications [26, 27, 13] have required the same averaged linear statistic, generalized so that $\log \lambda \mapsto \log(\lambda - z)$. In these references, methods involving integration over the unitary group have been used to find an explicit expression for this generalization of the RHS of (2.13) in terms of a $d \times d$ matrix. Thus with the eigenvalues of Σ^{-1} denoted as in Proposition 2 it is shown

$$\int_0^\infty \log(\lambda - z) \rho_{(1)}(\lambda; \Sigma) d\lambda = C_{k,d}(\{y_i\}) \sum_{m=1}^k \det[L_m] \quad (2.14)$$

where L_m is the $d \times d$ matrix with entries

$$(L_m)_{ij} = \begin{cases} \int_0^\infty \log(t - z) e^{-y_j t} t^{k-i} dt, & i = m \\ y_j^{-(k-i+1)} \Gamma(k-i+1), & i \neq m, i \leq k \\ y_j^{d-i}, & i > k \end{cases} \quad (2.15)$$

and

$$C_{k,d}(\{y_j\}) = \frac{(-1)^{k((k+1)/2-d)} \prod_{j=1}^d y_j^k}{\prod_{i=1}^{k-1} i! (\prod_{1 \leq i < j \leq d} (y_i - y_j))}. \quad (2.16)$$

For purposes of computing (1.9), and also for purpose of making the presentation more self contained, it is of interest to revise the derivation of (2.14). The first step is to consider $d \times k$ matrices X with a probability density function

$$P(X) = \frac{1}{\pi^{dk/2}} \det \Sigma^{-k} \exp(-\text{Tr}(XX^\dagger \Sigma^{-1})).$$

Changing variables to the nonzero eigenvalues and eigenvectors of XX^\dagger by writing $XX^\dagger = U \text{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0) U^\dagger$ for U unitary shows

$$P_k(\lambda_1, \dots, \lambda_k) = \frac{\det \Sigma^{-k}}{\prod_{l=0}^{k-1} \Gamma(2+l) \Gamma(d-k+1+l)} \prod_{l=1}^k \lambda_l^{d-k} \prod_{1 \leq j < l \leq k} (\lambda_j - \lambda_l)^2 \\ \times \lim_{\lambda_{k+1}, \lambda_d \rightarrow 0} \int \exp(-\text{Tr}(U \text{diag}(\lambda_1, \dots, \lambda_d) U^\dagger \Sigma^{-1})) (U^\dagger dU), \quad (2.17)$$

where $(U^\dagger dU)$ denotes the Haar volume form for $d \times d$ unitary matrices normalized so that $\int (U^\dagger dU) = 1$. The matrix integral is precisely what is known as the Harish-Chandra–Itzykson–Zuber integral (see e.g. [12, Prop. 11.6.1]), which can be evaluated as a determinant to give

$$P_k(\lambda_1, \dots, \lambda_k) = \frac{\det \Sigma^{-k} \prod_{l=0}^{d-1} \Gamma(1+l)}{\prod_{l=0}^{k-1} \Gamma(2+l) \Gamma(d-k+1+l)} \prod_{l=1}^k \lambda_l^{d-k} \prod_{1 \leq j < l \leq k} (\lambda_j - \lambda_l)^2$$

$$\times \lim_{\lambda_{k+1}, \lambda_d \rightarrow 0} \prod_{1 \leq j < l \leq d} \frac{(y_j - y_l)}{(\lambda_j - \lambda_l)} \det[e^{y_j \lambda_l}]_{j,l=1, \dots, d}. \quad (2.18)$$

The limit can be carried out by power series expanding columns $k+1, \dots, d$, leaving us with the explicit determinant formula

$$P_k(\lambda_1, \dots, \lambda_k) = \frac{(-1)^{k((k+1)/2-d)}}{\prod_{i=1}^k i!} \prod_{1 \leq j < l \leq k} (\lambda_j - \lambda_l) \frac{\prod_{j=1}^d y_j^k}{\prod_{1 \leq j < l \leq d} (y_j - y_l)}$$

$$\times \det \begin{bmatrix} [e^{-y_j \lambda_i}]_{i=1, \dots, k} \\ [y_j^{d-i}]_{i=d+1, \dots, k} \\ [y_j^{d-i}]_{i=d+1, \dots, d} \end{bmatrix} \quad (2.19)$$

The second step is to use (2.19) to compute the average

$$\left\langle \prod_{j=1}^k (\lambda_j - z)^\mu \right\rangle_{P_k} \quad (2.20)$$

For this, one notes that P_k consists of two anti-symmetric factors in $\{\lambda_j\}$. Since, according to the Vandermonde determinant formula

$$\prod_{1 \leq j < l \leq k} (\lambda_j - \lambda_l) = \mathcal{A} \prod_{l=1}^k \lambda_l^{k-l},$$

where \mathcal{A} denotes the anti-symmetrization operation, we can replace $\prod_{1 \leq j < l \leq k} (\lambda_j - \lambda_l)$ in the integrand implied by (2.20) by $k! \prod_{l=1}^k \lambda_l^{l-1}$. The integrals over $\{\lambda_j\}$ can now be done row-by-row in the remaining determinant, and we obtain

$$\left\langle \prod_{j=1}^k (\lambda_j - z)^\mu \right\rangle_{P_k} = \frac{(-1)^{k((k+1)/2-d)}}{\prod_{i=1}^{k-1} i!} \frac{\prod_{j=1}^d y_j^k}{\prod_{1 \leq j < l \leq d} (y_j - y_l)}$$

$$\times \det \begin{bmatrix} [\int_0^\infty (t-z)^\mu t^{k-i} e^{-y_j t} dt]_{i=1, \dots, k} \\ [y_j^{d-i}]_{i=k+1, \dots, d} \\ [y_j^{d-i}]_{i=k+1, \dots, d} \end{bmatrix}_{j=1, \dots, d}. \quad (2.21)$$

The third and final step is to differentiate this formula with respect to μ and set $\mu = 0$. On the LHS this gives the LHS of (2.14). Recalling that the differentiation of a

determinant with respect to a parameter is equal to the sum of determinants with a single in each differentiated, we see that the RHS of (2.14) indeed follows by performing this operation on the RHS of (2.21).

Setting $z = 0$ in (2.14) allows the integral in row $i = m$ to be evaluated. Doing this and also taking out appropriate common factors from each of the first k rows shows

$$\begin{aligned} & \int_0^\infty \log \lambda \rho_{(1)}(\lambda; \Sigma) d\lambda \\ &= \frac{(-1)^{k((k+1)/2-d)}}{\prod_{1 \leq i < j \leq d} (y_i - y_j)} \sum_{m=1}^k \det \begin{bmatrix} [y_j^{i-1}]_{\substack{i=1, \dots, m-1 \\ j=1, \dots, d}} \\ [-(\log y_j) y_j^{m-1} + y_j^{m-1} \Psi(k-i+1)]_{j=1, \dots, d} \\ [y_j^{d+k-i}]_{\substack{i=m+1, \dots, d \\ j=1, \dots, d}} \end{bmatrix}. \end{aligned} \quad (2.22)$$

Furthermore, reversing the order of the rows $i = k+1, \dots, d$ and recalling that in general a determinant with a single row having each entry a sum of two terms is equal to the sum of two determinants, it follows from (2.23) that

$$\begin{aligned} & \int_0^\infty \log \lambda \rho_{(1)}(\lambda; \Sigma) d\lambda = \frac{1}{\prod_{1 \leq i < j \leq d} (y_j - y_i)} \sum_{m=1}^k \\ & \left(- \det \begin{bmatrix} [y_j^{i-1}]_{\substack{i=1, \dots, m-1 \\ j=1, \dots, d}} \\ [(\log y_j) y_j^{m-1}]_{j=1, \dots, d} \\ [y_j^{d+k-i}]_{\substack{i=m+1, \dots, d \\ j=1, \dots, d}} \end{bmatrix} + \Psi(k-m+1) \det[y_j^{i-1}]_{i,j=1, \dots, d} \right). \end{aligned} \quad (2.23)$$

Substituting in (2.13) and making use of the Vandermonde determinant evaluation gives a result equivalent to (1.7).

2.4 Second proof of Corollary 1

Comparison of (1.7) and (1.8) shows that it suffices to check that

$$\frac{1}{\prod_{1 \leq i < j \leq d} (y_i - y_j)} \sum_{m=1}^d \det \begin{bmatrix} [y_j^{i-1}]_{\substack{i=1, \dots, m-1 \\ j=1, \dots, d}} \\ [(\log y_j) y_j^{m-1}]_{j=1, \dots, d} \\ [y_j^{i-1}]_{\substack{i=m+1, \dots, d \\ j=1, \dots, d}} \end{bmatrix} = \sum_{m=1}^d \log y_m. \quad (2.24)$$

For this purpose, we observe that the sum over determinants on the LHS is equal to

$$\frac{d}{d\mu} \det[y_j^{\mu+i-1}]_{i,j=1, \dots, d} \Big|_{\mu=0}.$$

But this expression in turn can be rewritten

$$\det[y_j^{i-1}]_{i,j=1, \dots, d} \frac{d}{d\mu} \prod_{j=1}^d y_j^\mu \Big|_{\mu=0} = \det[y_j^{i-1}]_{i,j=1, \dots, d} \sum_{m=1}^d \log y_m.$$

The identity (2.24) now follows from the Vandermonde determinant evaluation.

2.5 The sum rule (1.8) in the case of $A_i = \Sigma^{1/2} G_i^r$

Suppose the A_i in (1.1) are given by $A_i = \Sigma^{1/2} G_i^r$, where each G_i^r denotes a $d \times d$ real Gaussian with independent standard normals as entries. Let $\mathcal{N}_{d \times d}^r(0, 1)$ denote the set of real rectangular $d \times k$ Gaussian matrices in which each entry is a standard normal. Analogous to (2.5) we have

$$\mu_1 + \cdots + \mu_k = \left\langle \log \det \left(G_k^{rT} \Sigma G_k^r \right)^{1/2} \right\rangle_{\mathcal{N}_{d \times k}^r(0, 1)},$$

where G_k^r denotes G_k^r restricted to the first k columns. As in the argument below (2.10) in the complex case, for $d = k$ we can write $\det(G_k^{rT} \Sigma G_k^r)^{1/2} = \det(G_k^r G_k^{rT} \Sigma)^{1/2}$. Furthermore the probability density corresponding to $\mathcal{N}_{d \times k}^r(0, 1)$ is then proportional to $\exp(-\text{Tr } G_k^r G_k^{rT} / 2)$. Introducing the real Wishart matrix $W = G_d^r G_d^{rT}$, for which the corresponding Jacobian is proportional to $(\det W)^{-1/2}$, we therefore have

$$\mu_1 + \cdots + \mu_d = \frac{1}{2} \left(\log \det \Sigma + \left\langle (\det W)^{-1/2} \log(\det W) e^{-\text{Tr } W/2} \right\rangle_{W>0} \right),$$

where the average is over positive definite $k \times k$ real symmetric matrices. To evaluate this average we make use of (2.6) to write

$$\left\langle \log(\det W) e^{-\text{Tr } W/2} \right\rangle_{W>0} = \frac{d}{d\mu} \left\langle (\det W)^{\mu-1/2} e^{-\text{Tr } W/2} \right\rangle_{W>0} \Big|_{\mu=0},$$

Arguing now as in the derivation of (2.8) shows that the RHS is equal to

$$\frac{1}{\hat{Z}_{d,-1/2}} \frac{d}{d\mu} \hat{Z}_{d,-1/2+\mu} \Big|_{\mu=0}$$

where

$$\hat{Z}_{d,c} := \int_0^\infty dx_1 \cdots \int_0^\infty dx_d \prod_{j=1}^d e^{-x_j/2} x_j^c \prod_{1 \leq j < l \leq d} |x_l - x_j|.$$

Like (2.9), this integral has an evaluation in terms of a product of gamma functions (see e.g. [12, Prop. 4.7.3 with $\beta = 1$]), giving

$$\begin{aligned} \left\langle \log(\det W) e^{-\text{Tr } W/2} \right\rangle_{W>0} &= \frac{d}{d\mu} 2^{d\mu} \prod_{j=0}^{d-1} \frac{\Gamma(\mu + 1/2 + j/2)}{\Gamma(1/2 + j/2)} \Big|_{\mu=0}, \\ &= d \log 2 + \sum_{j=0}^{d-1} \Psi((j+1)/2). \end{aligned}$$

Consequently, the sought analogue of (1.8) is

$$\mu_1 + \cdots + \mu_d = \frac{1}{2} \left(\log \det \Sigma + d \log 2 + \sum_{j=0}^{d-1} \Psi((j+1)/2) \right). \quad (2.25)$$

2.6 The generalized maximum Lyapunov exponent

For the matrix norm in (1.9) we take $\|P_N\| = \sup_{\vec{x}:|\vec{x}|=1} |P_N \vec{x}|$. This gives

$$L(q) = \sup_{\vec{x}:|\vec{x}|=1} \log \left\langle |P_N \vec{x}|^q \right\rangle.$$

Arguing as in the derivation of (2.5), for the A_i in (1.1) given by (1.4) we have

$$L(q) = \log \left\langle \det \left(G_1^c \Sigma G_1^c \right)^{q/2} \right\rangle_{\mathcal{N}_{d \times 1}^c(0,1)}. \quad (2.26)$$

Now with P_k the probability density function of the nonzero eigenvalues for matrices XX^\dagger , with X an element of $\Sigma^{-1/2} \mathcal{N}_{d \times k}^c(0,1)$ as in (2.17)–(2.19), this can be rewritten

$$L(q) = \log \left\langle \lambda_1^{q/2} \right\rangle_{P_1}. \quad (2.27)$$

The average in (2.27) is given by (2.21) with $k = 1$, $z = 0$ and $\mu = q/2$, telling us that

$$L(q) = \log \left(\Gamma(1 + q/2) \frac{1}{\prod_{1 \leq j < l \leq d} (y_j - y_l)} \det \begin{bmatrix} [y_j^{-q/2}]_{j=1,\dots,d} \\ [y_j^{i-1}]_{i=2,\dots,d} \end{bmatrix} \right). \quad (2.28)$$

In the case that $\Sigma = I_d$, (2.26) is a function of $G_1^c \dagger G_1^c$, so a change of variables analogous to that used in the derivation of (2.7) gives

$$L(q) = \log \langle y^{q/2+d-1} e^{-y} \rangle_{y>0} = \log \frac{\Gamma(q/2 + d)}{\Gamma(d)}. \quad (2.29)$$

3 Lyapunov exponents for diffusing complex matrices

The Lyapunov exponents as defined below (1.2) relate to the dynamics of the linear system specified by the difference equation $\vec{x}_{i+1} = A_{i+1} \vec{x}_i$ for given \vec{x}_0 . As emphasized in [19], the continuous counterpart of this setting is the matrix stochastic differential equation

$$dX(t) = AX(t)dt + BX(t)dW(t)$$

where A and B are fixed $d \times d$ matrices, and $W(t)$ is a $d \times d$ matrix with complex Brownian entries.

In the case that A and B commute, one has

$$X(t) = \exp \left(\left(A - \frac{1}{2} B^2 \right) t + BW(t) \right) X(0).$$

Of particular interest is the case $A = \frac{1}{2} B^2$, $B = I$ so that $X(t) = \exp(W(t))X(0)$. Suppose furthermore that $W(t) = W_1(t) + iW_2(t)$ with $W_1(t)$ and $W_2(t)$ Hermitian matrices of complex Brownian motions. To specify the latter, let $\text{GUE}_d(0, \sigma)$ denote the probability

density on $d \times d$ Hermitian matrices H proportional to $\exp(-\text{Tr } H^2/\sigma^2)$. We then require that $W_j(1)$ has probability density proportional to $\text{GUE}_d(0, \sigma_j)$, for $j = 1, 2$. An analogous specification of $W(t)$ has been given in [19] for $W(t)$ consisting of real Brownian entries, and decomposed as $W(t) = S_1(t) + S_2(t)$, where $S_1(t)$ is symmetric, and $S_2(t)$ antisymmetric.

Generally the matrix $\exp W(1)$ can be constructed as

$$e^{W(1)} = \lim_{m \rightarrow \infty} e^{C_m(m)} e^{C_m(m-1)} \dots e^{C_m(1)}$$

where $C_m(j) := W(j/m) - W((j-1)/m)$. In the above specification of $W(t)$, $C_m(j)$ is independent of j and distributed as $C/m^{1/2}$, with C specified as in (1.10). Consequently $\exp W(1)$ has the distribution of A_i for A_i as specified by (1.10).

With this fact established, the argument leading to (2.5) can be used to show that for $\exp W(1)$

$$\mu_1 + \dots + \mu_k = \lim_{m \rightarrow \infty} m \left\langle \log \det \left(E_{d \times k}^T e^{C^\dagger/\sqrt{m}} e^{C/\sqrt{m}} E_{d \times k} \right) \right\rangle,$$

where the average is over matrices $C = H_1 + iH_2$, with $H_j \in \text{GUE}_d(0, \sigma_j)$ ($j = 1, 2$). Straightforward expansion in powers of $1/\sqrt{m}$ reduces the RHS to

$$\begin{aligned} & \frac{1}{2} \left\langle \text{Tr} \left(E_{d \times k}^T (C^\dagger + C)^2 E_{d \times k} - (E_{d \times k}^T (C^\dagger + C) E_{d \times k})^2 \right) \right\rangle \\ &= 2 \left\langle \text{Tr} \left(E_{d \times k}^T H_1^2 E_{d \times k} - (E_{d \times k}^T H_1 E_{d \times k})^2 \right) \right\rangle \\ &= 2 \left\langle \sum_{i=1}^k \sum_{j=k+1}^d |H_{ij}|^2 \right\rangle = \sigma_1^2 \sum_{j=1}^k (d-j). \end{aligned}$$

Consequently we have

$$\mu_k = \sigma_1^2 (d - 2k + 1) \quad (3.30)$$

(cf. [19, eq. (15)]). Note that this is independent of σ_2 , and that each Lyapunov exponent vanishes for $\sigma_1 = 0$, corresponding to $C = iH_2$. This latter point follows from $\exp W(1)$ then being a diffusion on $U(d)$, and so the modulus of the vectors is unchanged under the corresponding flow.

4 Discussion

Consider the case $k = 1$ of (2.1) and thus the maximal Lyapunov exponent. According to Proposition 2, for A_i given by (1.4), the exact value of the maximal Lyapunov exponent is

$$\begin{aligned} \mu_1 &= -\frac{1}{2 \prod_{1 \leq i < j \leq d} (y_j - y_i)} \det \begin{bmatrix} [\log y_j]_{j=1, \dots, d} \\ [y_j^{i-1}]_{i=2, \dots, d} \end{bmatrix} - \frac{1}{2} \gamma \\ &= -\frac{1}{2} \sum_{j=1}^d \frac{\log y_j}{\prod_{l=1, l \neq j}^d (1 - y_j/y_l)} - \frac{1}{2} \gamma, \end{aligned} \quad (4.1)$$

where γ denotes Euler's constant. In obtaining the first line the fact that $\Psi(1) = -\gamma$ has been used, while the second line follows from the first by expanding the determinant by the first row and using the Vandermonde determinant formula. Note that replacing $y_i \mapsto \sigma^{-2}y_i$ for each $i = 1, \dots, d$ changes μ_1 by $\mu_1 \mapsto \mu_1 + \log \sigma$. To see this from the first line in (4.1) requires using the Vandermonde determinant formula, while in the second line one requires the identity

$$\sum_{j=1}^d \frac{1}{\prod_{l=1, l \neq j}^d (1 - y_j/y_l)} = 1$$

(see e.g. [12, displayed equation below (4.153)]). This change to μ_1 is consistent with the corresponding mapping of the matrices $A_i \mapsto \sigma A_i$.

For given distinct $\{y_j\}$ we can use the second equation in (4.1) to give a numerical value of γ_1 . For example, with $d = 2$, $y_1 = 1$, $y_2 = 1/4$ we obtain $\mu_1 = \frac{4}{3} \log 2 - \gamma/2 = 0.63558 \dots$. In this case

$$A_i = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} G_i^c \quad (4.2)$$

where G_i^c is a 2×2 complex Gaussian matrix with entries standard complex normals. With $\vec{x}_0 = [1 \ 0]^T$, let us define $\vec{x}_i = A_i \vec{y}_{i-1}$ ($i = 1, 2, \dots$), where $\vec{y}_i := \vec{x}_i / |\vec{x}_i|$. For a given realization of $\{A_i\}$, it follows from (2.1) with $k = 1$ that

$$\mu_1 = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=1}^m \log |\vec{x}_j|. \quad (4.3)$$

Moreover, straightforward working shows that $\frac{1}{m} \sum_{j=1}^m \log |\vec{x}_j|$ has a Gaussian distribution with mean μ_1 and a standard deviation σ proportional to $1/m^{1/2}$,

$$\sigma^2 = \frac{1}{m} \left(\left\langle \left(\log \det \left(G_1^{c\dagger} \Sigma G_1^c \right)^{1/2} \right)^2 \right\rangle_{\mathcal{N}_{d \times 1}^c(0,1)} - \mu_1^2 \right)$$

This then provides a simple to implement Monte Carlo estimation of μ_1 [10]. In the present setting, with $m = 10^6$ we obtained the estimation $\mu_1 \approx 0.6341$.

Below (1.8) it was commented that for the average value of the sum of Lyapunov exponents to have a well defined limit for $d \rightarrow \infty$ it was necessary that the eigenvalues $\{y_m\}$ have the scaling form $y_m/d = Y(y_m/d)$ with $Y(0) = 0$. Under this circumstance it is well known (see e.g. [22, Th. 7.2.2]) that the eigenvalue distribution of $G^{c\dagger} \Sigma G^c$ tends to a well defined nonrandom limit with density $u_Y(t)$ say, supported on some interval $I \subset \mathbb{R}^+$. According to a result of Newman [18] (see also [14]), one then has $\lim_{d \rightarrow \infty} e^{\mu_1} = (\int_I t u_Y(t) dt)^{1/2}$. To derive this from (4.1) does not seem possible, although for a given $Y(x)$ (4.1) can be used to give a numerical estimation of μ_1 . For example, with $Y(x) = 1 + x$, computation of (4.1) with $d = 5000$ (using high precision arithmetic to avoid catastrophic cancellations) gives $\mu_1 \approx -0.183$.

The case $k = d$ of (2.1), corresponding to the smallest Lyapunov exponent, admits a form very similar to the second expression in (4.1). Thus expanding the determinant by

the final row and using the Vandermonde determinant evaluation gives

$$\mu_d = -\frac{1}{2} \sum_{j=1}^d \frac{\log y_j}{\prod_{l=1, l \neq j}^d (1 - y_l/y_j)} + \frac{1}{2} \Psi(d); \quad (4.4)$$

note the interchange of the indices in the denominator relative to (4.1). With $d = 2$, $y_1 = 1$, $y_2 = 1/4$ as corresponds to (4.2), this gives $\mu_2 = -\frac{1}{3} \log 2 + \frac{1}{2}(1 - \gamma) = -0.019656\dots$, and thus $\mu_1 + \mu_2 = \log 2 + 1/2 - \gamma = 0.615932\dots$ (note that this is consistent with (1.8)). Let $\vec{x}_0^{(1)} = [1 \ 0]^T$ and $\vec{x}_0^{(2)} = [0 \ 1]^T$, and define $\vec{x}_i^{(p)} = A_i \vec{y}_i^{(p)}$, ($p = 1, 2$, $i = 1, 2, \dots$) where $\{\vec{y}_i^{(1)}, \vec{y}_i^{(2)}\}$ is obtained from $\{\vec{x}_{i-1}^{(1)}, \vec{x}_{i-1}^{(2)}\}$ by the Gram-Schmidt orthonormalization procedure. For a given realization of $\{A_i\}$, the analogue of (4.3) is then

$$\mu_1 + \mu_2 = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=1}^m \log \det(Y_j^\dagger Y_j)^{1/2},$$

where Y_i is the 2×2 matrix with columns given by $\vec{y}_i^{(1)}$ and $\vec{y}_i^{(2)}$. This formula without the limit suggests a Monte Carlo estimation of $\mu_1 + \mu_2$ [10]. In the present setting, with A_i given by (4.2), and choosing $m = 10^6$ gave $\mu_1 + \mu_2 \approx 0.6146$.

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